Finite-dimensional representations of the quantum superalgebra $U_q(gl(3/2))$ in a reduced $U_q(gl(3/2))$ contains/implies $U_q(gl(3/1))$ contains/implies $U_q(gl(3))$ basis

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Finite-dimensional representations of the quantum superalgebra $U_q[gl(3/2)]$ in a reduced $U_q[gl(3/2)] \supset U_q[gl(3/1)] \supset U_q[gl(3)]$ basis

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Abstract. For generic q we give expressions for the transformations of all essentially typical finite-dimensional modules of the Hopf superalgebra $U_q[gl(3/2)]$. The latter is a deformation of the universal enveloping algebra of the Lie superalgebra $gl(3/2)$. The basis within each module is similar to the Gel'fand–Zetlin basis for $gl(5)$. We write down expressions for the transformations of the basis under the action of the Chevalley generators.

This paper is devoted to the study of a subclass of representations of the quantized associative superalgebra $U_q[gl(3/2)]$, which we call essentially typical (see definition 1). The motivation to consider this particular algebra stems, firstly, from the observation that it is relatively simple and nevertheless its representation theory remains so far undeveloped explicitly and, secondly, that it carries all the main features of the general $U_q[gl(n/m)]$ superalgebra. The hope is, therefore, that $U_q[gl(3/2)]$ could serve as a generic example to tackle as a next step the representation theory of the deformed superalgebra $gl(n/m)$.

To begin with we recall that a quantum algebra $U_q[G]$ associated with the Lie (super)algebra G is a deformation of the universal enveloping algebra $U[G]$ of G, preserving its Hopf algebra structure. The first example of this kind was found for $G = sl(2)$ [1] endowed with a Hopf structure given by Sklyanin [2]. An example of a quantum superalgebra, namely the LS $osp(1/2)$, was first considered by Kulish [3] and it was soon generalized to an arbitrary Kac–Moody superalgebra with a symmetrizable generalized Cartan matrix [4]; an independent approach for basic Lie superalgebras was given in [5–7].

Beginning with $U_q[osp(1/2)]$ [8] the representation theory of the quantum superalgebras also became a field of growing interest. Various oscillator representations of all infinite series of the basic Lie superalgebras have been found (see, for instance, [5,7,9,10]). Nevertheless the results in this respect are still modest, the main reason being that even in the non-deformed case the representation theory of the best known, namely the basic Lie superalgebras (LSs) is far from being complete. Even the problem of obtaining character formulae for all atypical modules is still not completely solved (see [11] and the references therein). Apart from the lowest-rank LSs, explicit expressions for the transformations of all finite-dimensional modules have been given so far only for the class $gl(n/1)$, $n = 1, 2, \ldots$ [12]. By explicit we understand what is usually meant in physics, namely introducing a basis in the representation space and writing down explicit expressions for the transformations

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of the basis under the action of the generators (or, equivalently, giving expressions for the matrix elements of the generators with respect to the selected basis). The results from [12] have been used in [13] in order to show that each finite-dimensional irreducible $gl(n/1)$ module can be deformed to an irreducible $U_q[gl(n/1)]$ module. In close analogy with the results of Jimbo [14] for $U_q[A_n]$, expressions were given for the transformations of the deformed $U_q[gl(n/1)]$ modules in terms of the undeformed Gel'fand–Zetlin basis. The present investigation is much along the same line. We deal, however, only with the representations we know explicitly, namely the essentially typical representations of $gl(3/2)$.

Before going to the deformed case, we recall that the Lie superalgebra (LS) $gl(3/2)$ is an extension of the basic LS $sl(3/2)$ [15] by a one-dimensional centre with generator I. A convenient basis in $gl(3/2)$ is given by the Weyl generators e_{ij}, $i, j = 1, 2, 3, 4, 5$, which satisfy the relations

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - (-1)^{\theta_{ij}\theta_{kl}}\delta_{il}e_{kj}\,. \tag{1}$$

Here and elsewhere in the text

$$[e_{ij}, e_{kl}] \equiv e_{ij}e_{kl} - (-1)^{\theta_{ij}\theta_{kl}}e_{kl}e_{ij} \tag{2}$$

and

$$\theta_{ij} = \theta_i + \theta_j \quad \theta_i = 0 \quad \text{for} \quad i = 1, 2, 3 \qquad \theta_i = 1 \quad \text{for} \quad i = 3, 4\,. \tag{3}$$

The Z_2-grading on $gl(3/2)$ is imposed from the requirement that e_{ij} is an even (respectively an odd) generator, if θ_{ij} is an even (respectively odd) number. The central element I reads

$$I = e_{11} + e_{22} + e_{33} + e_{44} + e_{55} \tag{4}$$

and $sl(3/2)$ is the factor algebra of $gl(3/2)$ with respect to the ideal generated by I. In the lowest 5×5 representation e_{ij} is a matrix with 1 on the intersection of the ith row and the jth column and zeros elsewhere, and I is the unit matrix.

The universal enveloping algebra $U[gl(3/2)]$ of $gl(3/2)$ is the free associative algebra of all generators e_{ij} with relations (1). A representation of the LS $gl(3/2)$ is by definition a representation of $U[gl(3/2)]$ viewed as a Z_2 (= {0, 1})-graded associative algebra, i.e. as associative superalgebra.

The irreducible finite-dimensional modules (= representation spaces) of any basic LS are either typical or atypical [16]. It is convenient to consider any irreducible $sl(3/2)$ module W as an irreducible $gl(3/2)$ module, setting I proportional to the identity operator in W. In this sense one can speak about typical or atypical $gl(3/2)$ modules. More precisely, by a typical (resp atypical) $gl(3/2)$ module we understand a $gl(3/2)$ module W, which remains typical (resp atypical) with respect to $sl(3/2)$ and for which I is proportional to the identity operator in W. A subclass of the typical modules will be relevant for us.

Definition 1 [17]. A typical $gl(3/2)$ module W is said to be essentially typical if it decomposes into a direct sum of only typical $gl(3/1)$ modules.

In the present paper we show that for generic values of q (i.e. q is not root of unity) any essentially typical $gl(3/2)$ module W can be deformed to an irreducible $U_q[gl(3/2)]$ module, which we also call essentially typical. We proceed to describe in some more detail a basis within each typical module (and, more generally, within a larger class of finite-dimensional induced modules, namely the Kac modules [18]), since the same basis will also be used in the deformed case.

Let $e_{11}, e_{22}, e_{33}, e_{44}, e_{55}$ be a basis in the Cartan subalgebra H and e^1, e^2, e^3, e^4, e^5 be the basis dual to it in H^*. Denote by $W([m]_5)$ the Kac module with highest weight:

$$[m]_5 = m_{15}e^1 + m_{25}e^2 + m_{35}e^3 + m_{45}e^4 + m_{55}e^5 \equiv [m_{15}, m_{25}, m_{35}, m_{45}, m_{55}] \tag{5}$$

and let

$$l_{i5} = m_{i5} - i + 4 \quad \text{for} \quad i = 1, 2, 3 \qquad \text{and} \qquad l_{i5} = -m_{i5} + i - 3 \quad \text{for} \quad i = 4, 5\,. \tag{6}$$

Proposition 1 [17]. The Kac modules $W([m]_5)$ are in one-to-one correspondence with the set of all complex coordinates of the highest weight, which satisfy the conditions

$$m_{15} - m_{25}, m_{25} - m_{35}, m_{45} - m_{55} \in \mathbf{Z}_+ \qquad (= \text{all non-negative integers})\,. \tag{7}$$

The module $W([m]_5)$ is typical if and only if all $l_{15}, l_{25}, l_{35}, l_{45}, l_{55}$ are different. The module $W([m]_5)$ is essentially typical if and only if $l_{15}, l_{25}, l_{35} \neq l_{45}, l_{45}+1, l_{45}+2, \ldots, l_{55}$.

Let (m) be a pattern consisting of 15 complex numbers m_{ij}, $i \leqslant j = 1, 2, 3, 4, 5$, ordered as in the usual Gel'fan–Zetlin basis for $gl(5)$ [19], i.e.

$$(m) \equiv \begin{pmatrix} m_{15} & m_{25} & m_{35} & m_{45} & m_{55} \\ m_{14} & m_{24} & m_{34} & m_{44} & \\ m_{13} & m_{23} & m_{33} & & \\ m_{12} & m_{22} & & & \\ m_{11} & & & & \end{pmatrix}. \tag{8}$$

Proposition 2 [20]. The set of all patterns (8), whose entries satisfy the conditions

(1) m_{15}, m_{25}, m_{35}, m_{45}, m_{55} are fixed and the same for all patterns
(2) for each $i = 1, 2, 3$ and $p = 4, 5$, $m_{ip} - m_{i,p-1} \equiv \theta_{i,p-1} \in \mathbf{Z}_2$
(3) for each $i = 1, 2$ and $p = 4, 5$, $m_{ip} - m_{i+1,p} \in \mathbf{Z}_+$ (9)
(4) for each $i \leqslant j = 1, 2$ $m_{i,j+1} - m_{ij}$, $m_{ij} - m_{i+1,j+1} \in \mathbf{Z}_+$
(5) $m_{45} - m_{44}$, $m_{44} - m_{55} \in \mathbf{Z}_+$ constitute a basis in the Kac module $W([m]_5)$, which we refer to as a GZ-basis.

The quantum superalgebra $U_q[gl(3/2)]$ we consider here is a one-parameter Hopf deformation of the universal enveloping algebra of $gl(3/2)$ with a deformation parameter $q = e^{\hbar}$. As usual, the limit $q \to 1$ ($\hbar \to 0$) corresponds to the non-deformed case. More precisely, $U_q[gl(3/2)] \equiv U_q$ is a free associative algebra with unity, generated by e_i, f_i and $k_j \equiv q^{h_j/2}$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, 4, 5$, which satisfy the following (unless otherwise stated the indices below run over all possible values).

(1) The Cartan relations

$$k_i k_j = k_j k_i \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1 \tag{10}$$

$$k_i e_j k_i^{-1} = q^{1/2(\delta_{ij} - \delta_{i,j+1})} e_j \qquad k_i f_j k_i^{-1} = q^{-1/2(\delta_{ij} - \delta_{i,j+1})} f_j \tag{11}$$

$$e_i f_j - f_j e_i = 0 \quad i \neq j \qquad e_i f_i - f_i e_i = (q - q^{-1})^{-1}(k_i^2 k_{i+1}^{-2} - k_{i+1}^2 k_i^{-2}) \quad i = 1, 2, 4 \tag{12}$$

$$e_3 f_3 + f_3 e_3 = (q - q^{-1})^{-1}(k_3^2 k_4^2 - k_3^{-2} k_4^{-2})\,. \tag{13}$$

(2) The Serre relations for the positive simple root vectors (E-Serre relations)

$$e_ie_j = e_je_i \quad |i-j| \neq 1 \qquad e_3^2 = 0 \tag{14}$$

$$e_i^2e_{i+1} - (q+q^{-1})e_ie_{i+1}e_i + e_{i+1}e_i^2 = 0 \qquad i = 1,2 \tag{15}$$

$$e_{i+1}^2e_i - (q+q^{-1})e_{i+1}e_ie_{i+1} + e_ie_{i+1}^2 = 0 \qquad i = 1,3 \tag{16}$$

$$e_3e_2e_3e_4 + e_2e_3e_4e_3 + e_3e_4e_3e_2 + e_4e_3e_2e_3 - (q+q^{-1})e_3e_2e_4e_3 = 0\,. \tag{17}$$

(3) The relations obtained from (14)–(17) by replacing everywhere e_i by f_i (F-Serre relations).

Equation (17) is the extra Serre relation, discovered recently [21–23] and reported by Tolstoy *et al* [24].

The Z_2-grading on $U_q[gl(3/2)]$ is defined from the requirement that the only odd generators are e_3 and f_3. U_q is a Hopf algebra with a co-unity ε, a comultiplication Δ and an antipode S defined as

$$\varepsilon(e_i) = \varepsilon(f_i) = \varepsilon(k_i) = 0 \tag{18}$$

$$\begin{aligned} &\Delta(k_i) = k_i \otimes k_i \\ &\Delta(e_i) = e_i \otimes k_ik_{i+1}^{-1} + k_i^{-1}k_{i+1} \otimes e_i \quad i \neq 3 \qquad \Delta(e_3) = e_3 \otimes k_3k_4 + k_3^{-1}k_4^{-1} \otimes e_3 \\ &\Delta(f_i) = f_i \otimes k_ik_{i+1}^{-1} + k_i^{-1}k_{i+1} \otimes f_i \quad i \neq 3 \qquad \Delta(f_3) = f_3 \otimes k_3k_4 + k_3^{-1}k_4^{-1} \otimes f_3 \end{aligned} \tag{19}$$

$$S(k_i) = k_i^{-1} \; S(e_j) = -qe_j \; S(f_j) = -q^{-1}f_j \quad j \neq 3 \qquad S(e_3) = -e_3 \; S(f_3) = -f_3\,. \tag{20}$$

Now we are ready to state our main result. To this end denote by $(m)_{\pm ij}$ a pattern obtained from the GZ pattern (m) after a replacement of m_{ij} with $m_{ij} \pm 1$ and let

$$\begin{aligned} &l_{ij} = m_{ij} - i + 4 \quad \text{for} \quad i = 1,2,3 \qquad \text{and} \qquad l_{ij} = -m_{ij} + i - 3 \quad \text{for} \quad i = 4,5 \\ &[x] = \frac{q^x - q^{-x}}{q - q^{-1}}\,. \end{aligned} \tag{21}$$

Proposition 3. For generic values of q every essentially typical $gl(3/2)$ module $W([m]_5)$ can be deformed to an irreducible $U_q[gl(3/2)]$ module. The transformations of the basis under the action of the algebra generators read

$$k_i(m) = q^{\frac{1}{2}(\sum_{j=1}^{i} m_{ji} - \sum_{j=1}^{i-1} m_{j,i-1})}(m) \qquad i = 1,2,3,4,5 \tag{22}$$

$$e_k(m) = \sum_{j=1}^{k}\left(-\frac{\prod_{i=1}^{k+1}[l_{i,k+1} - l_{jk}]\prod_{i=1}^{k-1}[l_{i,k-1} - l_{jk} - 1]}{\prod_{i\neq j=1}^{k}[l_{ik} - l_{jk}][l_{ik} - l_{jk} - 1]}\right)^{1/2}(m)_{jk} \qquad k = 1,2 \tag{23}$$

$$f_k(m) = \sum_{j=1}^{k}\left(-\frac{\prod_{i=1}^{k+1}[l_{i,k+1} - l_{jk} + 1]\prod_{i=1}^{k-1}[l_{i,k-1} - l_{jk}]}{\prod_{i\neq j=1}^{k}[l_{ik} - l_{jk} + 1][l_{ik} - l_{jk}]}\right)^{1/2}(m)_{-jk} \qquad k = 1,2 \tag{24}$$

$$e_3(m) = \sum_{i=1}^{3}\theta_{i3}(-1)^{i-1}(-1)^{\theta_{13}+\cdots+\theta_{i-1,3}}\left(\frac{\prod_{k=1}^{2}[l_{k2} - l_{i4}]}{\prod_{k\neq i=1}^{3}[l_{k4} - l_{i4}]}\right)^{1/2}(m)_{i3} \tag{25}$$

$$f_3(m) = \sum_{i=1}^{3}(1-\theta_{i3})(-1)^{i-1}(-1)^{\theta_{13}+\cdots+\theta_{i-1,3}}[l_{i4} - l_{44}]\left(\frac{\prod_{k=1}^{2}[l_{k2} - l_{i4}]}{\prod_{k\neq i=1}^{3}[l_{k4} - l_{i4}]}\right)^{1/2}(m)_{-i3} \tag{26}$$

$$e_4(m) = \sum_{i=1}^{3} \theta_{i4}(1-\theta_{i3})(-1)^{\theta_{14}+\cdots+\theta_{i-1,4}+\theta_{i+1,3}+\cdots+\theta_{33}} \prod_{k\neq i=1}^{3} \left(\frac{[l_{i5}-l_{k4}][l_{i5}-l_{k4}-1]}{[l_{i5}-l_{k5}][l_{i5}-l_{k3}-1]} \right)^{1/2} (m)_{i4}$$
$$+ ([l_{44}-l_{45}][l_{55}-l_{44}])^{1/2} \prod_{k=1}^{3} \frac{[l_{k4}-l_{44}][l_{k4}-l_{44}+1]}{[l_{k5}-l_{44}][l_{k3}-l_{44}+1]} (m)_{44} \tag{27}$$

$$f_4(m) = ([l_{44}-l_{45}+1][l_{55}-l_{44}-1])^{1/2}(m)_{-44} + \sum_{i=1}^{3} \theta_{i3}(1-\theta_{i4})(-1)^{\theta_{14}+\cdots+\theta_{i-1,4}+\theta_{i+1,3}+\cdots+\theta_{33}}$$
$$\times \frac{[l_{i5}-l_{45}][l_{i5}-l_{55}]}{[l_{i5}-l_{44}-1][l_{i5}-l_{44}]} \prod_{k\neq i=1}^{3} \left(\frac{[l_{i5}-l_{k4}][l_{i5}-l_{k4}-1]}{[l_{i5}-l_{k5}][l_{i5}-l_{k3}-1]} \right)^{1/2} (m)_{-i4}\,. \tag{28}$$

The proof is straightforward but lengthy. In order to perform it one has to check that the Cartan relations and the E- and F-Serre relations hold as operator equations in $W([m]_5)$. The irreducibility follows from the results of Zhang [25]. It is not difficult to check it directly and follows from the observation that for generic q a deformed matrix element in the GZ basis is zero only if the corresponding non-deformed matrix element vanishes [see (22)–(28)].

Remark. If a vector from the RHS of (22)–(28) does not belong to the module under consideration, then it is assumed that the corresponding term is zero even if the coefficient in front of it is undefined. If an equal number of factors in a numerator and a denominator are simultaneously equal to zero, then one has to cancel them out.

The operators $e_{i-1},\ f_{i-1},\ k_{i-1},\ k_i,\ i = 1, 2, \ldots, n$ generate a chain of Hopf subalgebras $U_q[gl(1)]$ for $n = 1$, $U_q[gl(2)]$ for $n = 2$, $U_q[gl(3)]$ for $n = 3$, $U_q[gl(3/1)]$ for $n = 4$, $U_q[gl(3/2)]$ for $n = 5$:

$$U_q[gl(1)] \subset U_q[gl(2)] \subset U_q[gl(3)] \subset U_q[gl(3/1)] \subset U_q[gl(3/2)]\,. \tag{29}$$

The row $[m]_n \equiv (m_{1n}, \ldots, m_{1n})$ in a GZ pattern (m) indicates that (m) belongs to an irreducible submodule $W([m]_n) \subset W([m]_5)$ of the subalgebra with the corresponding n. Therefore the decomposition of any essentially typical module $W([m]_5)$ along the chain (29) is evident and the GZ basis is reduced with respect to this chain. More precisely (as this is the case also with the GZ basis of $gl(5)$), each basis vector (m) is an element from one and only one chain of submodules:

$$(m) \in W([m]_1) \subset W([m]_2) \subset W([m]_3) \subset W([m]_4) \subset W([m]_5)\,. \tag{30}$$

Unfortunately we do not know at present how to extend the above results beyond the class of the essentially typical representations, although it is known that each irreducible finite-dimensional representation can be deformed to a $U_q[gl(3/2)]$ representation [24]. We hope, however, that the present approach could at least be generalized for the essentially typical representations of $U_q[gl(n/m)]$, although technically this could be quite difficult. It still remains an open question whether one can deform the Kac modules even for generic q.

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